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Reg. No. : ....

Code No.: 6022 **Sub. Code: PMAM 41**

M.Sc. (CBCS) DEGREE EXAMINATION, NOVEMBER 2022

Fourth Semester

Mathematics

FUNCTIONAL ANALYSIS

(For those who joined in July 2017-2020 onwards)

Time : Three hours Maximum : 75 marks

PART A — ($10 \times 1 = 10$ marks)

Answer ALL questions.

Choose the correct answer :

1. The normed linear space N is a ________ space with respect to the metric d defined by $d(x,y)=\|x-y\|$

   (a) Metric (b) Complete

   (c) Hilbert (d) Inner

2. The spaces R and C the real numbers and the complex numbers are the simplest of all ________ spaces

   (a) Complex (b) Hilbert

   (c) Normed linear (d) None

3. The conjugate' space of N* is called as ________conjugate.

   (a) second (b) dual of N

   (c) third (d) first

4. The isometric isomorphism $x \to F_x$ is called the ______of N into N**.

   (a) banach (b) natural imbedding

   (c) surjective (d) injuctive

5. A ________ space is a complex banach space whose norm arises from the inner product.

   (a) Hilbert (b) Banach

   (c) Inner product (d) Banach algebra

6. A ________set in a Hilbert space H is a non empty subset of H which consists of mutually orthogonal unit vectors.

   (a) Hilbert (b) Empty

   (c) Orthonormal (d) Banach

7. The value of T^{**} = ________.

   (a) T (b) T^*

   (c) T^{1} (d) T_1

8. The operator T is self adjoint if A = ________.

   (a) A (b) A^2

   (c) A^* (d) A^{**}

9. The value of det(1) = ________.

   (a) 0 (b) 1

   (c) -1 (d) 2

10. $\det(T) \neq 0$ if and only if T is ________.

    (a) singular (b) unitary

    (c) non singular (d) self adjoint

PART B — ($5 \times 5 = 25$ marks)

Answer ALL questions, choosing either (a) or (b).

11. (a) Prove that if M is a closed linear subspace of a normed linear space N and x_0 is a vector not in M, then there is a functional f_0 in N^* such that $f_0(M)=0$ and $f_0(x_0)\neq 0$.

    Or

    (b) Let N and N' be normed linear spaces and T a linear transformation of N into N'. Then prove that the following are equivalent.

    (i) T is continuous

    (ii) T is continuous at the origin

    (iii) There exists a real number $K \geq 0$ with the property that $\|T(x)\| \leq K\|x\|$ for every x in N.

    (iv) If $S\{x : \|x\| \leq 1\}$ is the closed unit sphere in N, then its image $T(S)$ is a bounded set in N'.

12. (a) Prove that if n is a normed linear space then the closed unit sphere S^* in N^* is a compact Hausdorff space in the weak* topology.

    Or

    (b) Prove that if B and B' are Banach spaces and if T is a linear transformation of B into B' then T is continuous if and only if its graph is closed.

13. (a) Prove that if M is a proper closed linear subspace of a Hilbert space H, then there exists a non zero vector z_0 in H such that $z_0 \perp M$.

    Or

    (b) Prove that if x and y are any two vectors in a Hilbert space H, then $|\langle x, y \rangle| \leq \|x\|\|y\|$.

14. (a) The adjoint operation $T \to T^*$ on B(H) has the following properties Prove them

(i) $(T_1 + T_2)^* = T_1^* + T_2^*$

(ii) $(\alpha T)^* = \bar{\alpha} T^*$

(iii) $(T_1 T_2)^* = T_2^* T_1^* T^{**} = T$

Or

(b) Prove that if T is an operator on H, for which $(Tx, x) = 0$ for all x then $T = 0$.

15. (a) Prove that an operator T on H is unitary if and only if it is an isometric isomorphism of H onto itself.

Or

(b) If T is normal, then prove that the Mi's are pairwise orthogonal.

PART C — (5 × 8 = 40 marks)

Answer ALL questions, choosing either (a) or (b).

16. (a) Let M be a closed linear space of a normed linear space N. If the norm of a coset $x+M$ in the quotient space N/M is defined by $\|x + M\| = \inf\{\|x + m\| : m \in M\}$ then prove that N/M is a normed linear space.

Or

(b) State and prove Hahn banach theorem.

17. (a) Prove that if B and B' are Banach spaces, and if T is a continuous linear transformation of B onto B', then T is an open mapping.

Or

(b) If T is an operator on a nls N, then prove that its conjugate T^* defined by $[T^*(f)](x) = f(T(x))$ is an operator on N^* and the mapping $T \to T^*$ is an isometric isomorphism of $\mathfrak{B}(N^*)$ into $\mathfrak{B}(N^*)$ which reverses products and preserves the identity transformation.

18. (a) Prove that if M and N are closed linear subspaces of a Hubert space H such that $M \perp N$, then the linear subspace $M + N$ is also closed.

Or

(b) Prove that a closed convex subset C of a Hilbert space H contains a unique vector of smallest norm.

19. (a) Let H be a Hilbert space, and let f be an arbitrary functional in H^*, then prove that there exists a unique vector y in H such that $f(x) = (x, y)$ for every x in H.

Or

(b) Prove that if $\{e_i\}$ is an ortho normal set in a Hilbert space H, and if x is an arbitrary vector in H then $x - \sum (x, e_i) e_i \perp e_j$ for each j.

20. (a) Prove that if $B = \{e_i\}$ is a basis for H, then the mapping $T \to [T]$, which assigns to each operator T its matrix relative to B, is an isomorphism of the algebra $\mathfrak{B}(H)$ onto the total matrix algebra A_n.

Or

(b) (i) Prove that $\|N^2\| = \|N\|^2$ if N is normal operator on H.

(ii) Also prove that if T is an operator on H, then T is normal iff its real and imaginary parts commute.